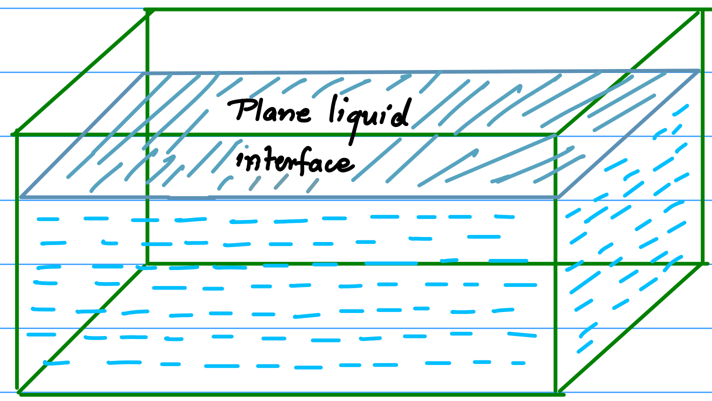
Static equilibrium of liquid interfaces

Surface tension contributes to the balance of forces on the fluid at the interface. We will consider the configuration of static interfaces in the presence of surface tension.

Planar liquid interface

Consider a patch of a planar liquid interface as shown in Figure 1.



The patch could be made of the liquid interface itself or could be another material. It could also be a thin slice of a floating body.

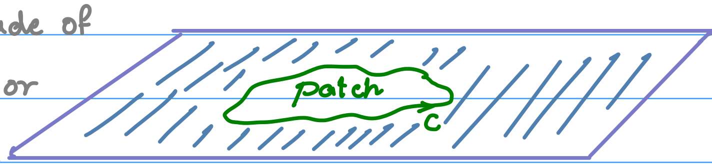


Figure 1. Schematic of an interfacial patch.

Let the boundary of the patch be denoted by the curve C. The force of surface tension $\underline{F}$ on this patch is:

$$\underline{F} = \int_C \sigma \, \hat{n} \, dl \quad \text{(from Introduction to surface tension)}$$

- dl – infinitesimal line element
- $\hat{n}$ – unit normal to the curve C, but tangent to the interface
- σ – Coefficient of surface tension A.K.A. surface tension A.K.A interfacial tension.
- C – Curve describing the boundary of the portion

Claim: If σ = constant, then $\underline{F} = \underline{0}$. (Hint: The interface is planar.)

Proof : $\underline{F} = \int_C \sigma \hat{n}\, dl$ applying divergence theorem over the patch A,

$= \int_A \nabla\sigma \, dA$ ... (Equation 1)

If $\sigma =$ constant, then $\nabla\sigma \equiv 0$, and hence $\underline{F} = 0$.

A unbalanced force can exist on a planar interface only if the surface tension is not constant. This unbalanced force is necessarily tangential to the surface.

Although the force of surface tension is proportional to length of the curve C
When surface tension is not constant, the infinitesimal unbalanced force on an infinitesimal patch is not proportional to the circumference of the patch but the area of the patch.

$dF = \nabla\sigma\, dA$ ... from (Equation 1).

Thus, we have the manifestation of a quantity which behaves like a stress (force proportional to area $\Rightarrow$ force / area = const.)

This quantity $\nabla\sigma$ is called the Marangoni stress, after the Italian physicist Carlo Marangoni (1840 – 1925).

Gradients of surface tension may exist on the interface due to non-uniform temperature, solvents, or surfactants (like soap).

More about Marangoni stresses later in the course. Let's focus now on the case $\nabla\sigma = 0$ or $\sigma =$ constant. There could be unbalanced forces if the interface is curved.

Curved Liquid Interface

For a curved interface, the force balance also has a component normal to the interface $\hat{N}$.

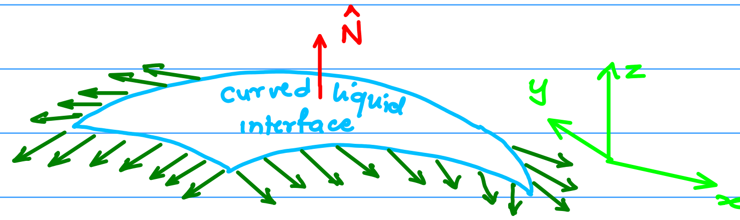


Figure 2. Forces on a patch of curved interface. $\hat{N} = \hat{z}$

$$\underline{F} = \int_c \sigma \, \hat{n} \, dl \quad \ldots \text{ (from Introduction to surface tension)}$$

Consider an infinitesimal patch of the curved interface as shown in Figure 2. Because σ = constant, it can be moved out of the integral.
The unbalanced force may be easily estimated using a rectangular patch.

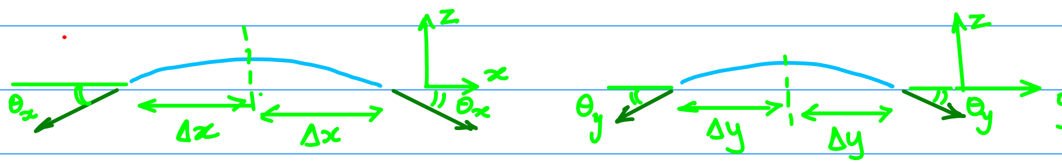


Figure 2. Two perspectives of the infinitesimal interface patch.

Two perspectives of this patch are shown in Figure 2. The unbalanced force may be written as

$$\underline{F} = 2\sigma \Delta y \left[\cancel{-\cos\theta_x \hat{x}} - \sin\theta_x \hat{z} + \cancel{\cos\theta_x \hat{x}} - \sin\theta_x \hat{z} \right]$$
$$+ 2\sigma \Delta x \left[\cancel{-\cos\theta_y \hat{y}} - \sin\theta_y \hat{z} + \cancel{\cos\theta_y \hat{y}} - \sin\theta_y \hat{y} \right]$$

$$= -4\sigma \left[\Delta y \sin\theta_x + \Delta x \sin\theta_y \right] \hat{z} \ \ldots$$ unbalanced force is only in the direction normal to the interface.

Significant further simplification is possible if we realize that if

κ_{xx} = curvature along x

κ_{yy} = curvature along y.

then $\theta_x = \kappa_{xx} \Delta x$, $\theta_y = \kappa_{yy} \Delta y$ by definition of curvature, and since Δx and Δy are infinitesimal, so are θ_x and θ_y.

Hence we can simplify $\sin\theta_x \approx \theta_x = \kappa_{xx}\Delta x$

$\sin\theta_y \approx \theta_y = \kappa_{yy}\Delta y$

$$\Rightarrow \underline{F} = 4\sigma\,\Delta x\,\Delta y\,(\kappa_{xx}+\kappa_{yy})\,\hat{z} = \sigma(\kappa_{xx}+\kappa_{yy})\,\Delta A\,\hat{z}.$$ ΔA = area of the patch.

Again, we find that although the force of surface tension acts per unit length, the unbalanced force on an infinitesimal patch is proportional to the area of the patch. The constant of proportionality is called the Laplace stress or Laplace pressure (after the French mathematician and astronomer Pierre-Simon Laplace (1749-1827)).

Laplace pressure $p_{Laplace} = \sigma(\kappa_{xx}+\kappa_{yy})$.

The quantity $\dfrac{\kappa_{xx}+\kappa_{yy}}{2} = \kappa_{mean}$, is called the mean curvature of the surface. In fact, κ_{mean} does not depend on the choice of x and y axis as long as they are (i) tangential to the surface, and (ii) perpendicular to each other.

$$\Rightarrow \boxed{p_{Laplace} = 2\sigma\kappa_{mean},} \quad \text{and} \quad \boxed{\underline{F} = p_{Laplace}\,\hat{N} = 2\sigma\kappa_{mean}\,\hat{N}\,dA.}$$

Consequences of Laplace pressure

(i) Static shape of a blob of liquid in the absence of gravity

From hydrostatics, in the absence of gravity, the pressure in the drop must be constant $= p_{drop}$.

A force balance on a patch of the interface in the normal direction gives

$$p_{drop}\,dA = p_{atm}\,dA + p_{Laplace}\,dA,$$

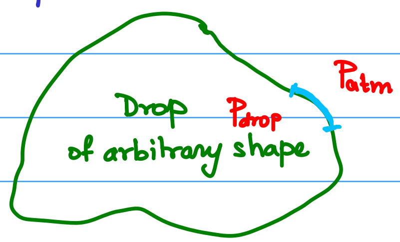


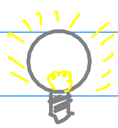
Figure 3.

at every point on the surface of the drop. Since $P_{drop} = \text{const}$ and $P_{atm} = \text{const}$ $\Rightarrow$ $P_{Laplace}$ must also be constant.

$$\Rightarrow \quad 2\sigma \kappa_{mean} = \text{constant}$$

$$\Rightarrow \quad \kappa_{mean} = \text{constant}.$$

Hence the mean curvature of the interface of a blob of liquid is constant. The pressure inside the drop is $P_{drop} = P_{atm} + 2\sigma\kappa_{mean}$.



An obvious example of a constant mean curvature surface is a sphere. Small drops also attain a spherical shape because the variation of hydrostatic pressure inside the drop due to gravity is negligible (compared to the Laplace pressure).

Can you think of other constant mean curvature surfaces? How about zero mean curvature surfaces?

Question: How small must a drop be to qualify as a small drop (i.e. for gravity to be negligible)?

Answer: Consider drop of radius R. Hydrostatic pressure variation in the drop due to gravity is $\rho g (2R)$. Laplace pressure inside the drop is $2\sigma\kappa_{mean} = 2\sigma/R$. For the hydrostatic pressure variation to be negligible $\rho g \, 2R \ll \frac{2\sigma}{R}$ or $R \ll \left(\frac{\sigma}{\rho g}\right)^{1/2}$.

The scale $\left(\frac{\sigma}{\rho g}\right)^{1/2}$ is the characteristic length that shows up over and over again when gravity is compared with surface tension. Hence, it is given a special name: Capillary length $= \left(\frac{\sigma}{\rho g}\right)^{1/2}$.