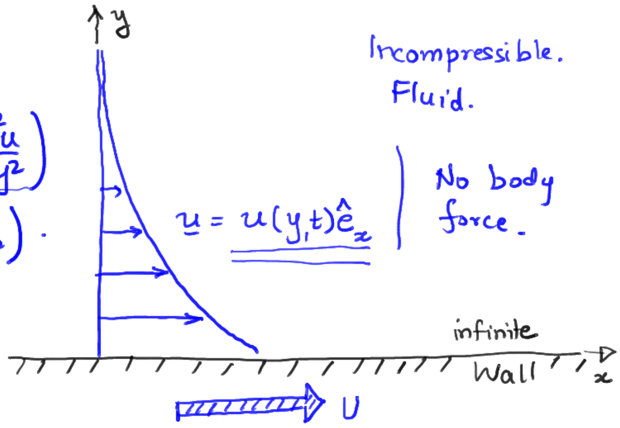
UNSTEADY ONE-DIMENSIONAL FLOWS.

- Exploit one-dimensional character for solving unsteady flows
- Unsteady versions of plane, cylindrical and circular Couette-Poiseuille flows.
- Boundary driven flows in semi-infinite domains (Stokes problems)

STOKES FIRST PROBLEM



Incompressible Fluid.

No body force.

x: $$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \cancelto{0}{\rho g_x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

y: $$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{\partial p}{\partial y} + \cancelto{0}{\rho g_y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right).$$

mass: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Substituting the one-dimensional profile.

x-momentum: $$\rho\frac{\partial u}{\partial t} = -\cancelto{0}{\frac{\partial p}{\partial x}} + \mu\frac{\partial^2 u}{\partial y^2}$$

y-momentum: $$0 = -\frac{\partial p}{\partial y} \Rightarrow p(x,t)$$

mass : $0 = 0$ ... satisfied trivially.

As $y\to\infty$, $u\to 0 \Rightarrow \frac{\partial p}{\partial x}\to 0$.

$\Rightarrow \frac{\partial p}{\partial x} = 0$ everywhere!

$p(t)$ alone.

Partial differential equation to solve. for $u(y,t)$

$\left(\nu = \frac{\mu}{\rho}\right)$ $$\frac{\partial u}{\partial t} = \nu\,\frac{\partial^2 u}{\partial y^2} \qquad\qquad \frac{\partial \tilde{u}}{\partial \tilde{t}} = \nu\frac{\alpha}{\beta^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

Transform $t = \alpha\tilde{t}$, $y = \beta\tilde{y}$, $u = \gamma\tilde{u}$

$$\frac{\partial}{\partial t} = \frac{\partial \tilde{t}}{\partial t}\frac{\partial}{\partial \tilde{t}} = \frac{1}{\alpha}\frac{\partial}{\partial \tilde{t}}.$$

$$\frac{\partial}{\partial y} = \frac{1}{\beta}\frac{\partial}{\partial \tilde{y}}$$

$u = 0$ @ $t = 0$ $\xrightarrow{t=\alpha\tilde{t},\ y=\beta\tilde{y},\ u=\gamma\tilde{u}}$ $\tilde{u} = 0$ @ $\tilde{t} = 0$

$u = 0$ @ $y\to\infty$ $\qquad$ $\tilde{u} = 0$ @ $\tilde{y} = \infty$

$u = U$ @ $y = 0,\ t>0$ $\longrightarrow$ $\tilde{u} = \frac{U}{\gamma}$ @ $\tilde{y} = 0,\ \tilde{t}>0$.

$\underbrace{\text{The equations are invariant}}$ if $\underbrace{\alpha = \beta^2,\ \gamma = 1}$.

If $u(y,t) = f(y,t)$ is a solution THEN $\tilde{u}(\tilde{y},\tilde{t}) = f(\tilde{y},\tilde{t})$ must be a solution.

$$\boxed{u(y,t) = \frac{1}{\gamma}u\left(\frac{y}{\beta},\frac{t}{\alpha}\right)}$$ for arbitrary $\alpha = \beta^2 > 0$.

Choose $\alpha = t$, $\beta = \sqrt{t}$ and $$\boxed{u(y,t) = \frac{1}{1}u\left(\frac{y}{\sqrt{t}},1\right)}$$ ... self similar profile.

Similarity solution :

Step 1 — Scaling analysis; Convert derivatives to fractions casually

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \longrightarrow \frac{u}{t} = \frac{\nu u}{y^2} \Rightarrow y = \sqrt{\nu t}.$$

Define new variables accordingly.

$u = 0$ @ $t=0$

$u = 0$ @ $y = \infty$

$u = U$ @ $y=0, t>0$. $\longrightarrow u = U$.

$$\frac{u}{U} = f(\xi), \quad \xi = \frac{y}{\delta(t)}$$

Step 2:

Now make a careful substitution

$$\delta = 2\sqrt{\nu t}$$

$$\frac{d\delta}{dt} = \frac{2}{2}\sqrt{\frac{\nu}{t}} = \frac{\delta}{2t}.$$

$$\frac{\partial \xi}{\partial t} = -\frac{y}{\delta^2} \cdot \frac{d\delta}{dt} = -\frac{y}{\delta^2} \cdot \frac{\delta}{2t} = -\frac{y}{\delta} \cdot \frac{1}{2t} = -\frac{\xi}{2t}.$$

$$\frac{\partial \xi}{\partial y} = \frac{1}{\delta}$$

$$\frac{\partial u}{\partial t} = U f'(\xi) \frac{\partial \xi}{\partial t} = -\frac{U\xi}{2t} f'(\xi)$$

$$\frac{\partial u}{\partial y} = U \frac{\partial}{\partial y} f(\xi) = \frac{U f'(\xi)}{\delta}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U}{\delta^2} f''(\xi)$$

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \longrightarrow -\frac{U\xi f'(\xi)}{2t} = \frac{\nu U}{\delta^2} f''(\xi).$$

note. $\boxed{\frac{1}{4t} = \frac{\nu}{\delta^2}}$

$$\Rightarrow \boxed{f''(\xi) + 2\xi f'(\xi) = 0}$$ ODE for $f(\xi)$.

Initial and boundary conditions:

$$f(\xi \to \infty) = 0$$
$$f(\xi = 0) = 1.$$

Solution: $\frac{d}{d\xi}\left[f'(\xi)\, e^{\xi^2}\right] = 0. \Rightarrow f'(\xi) = A e^{-\xi^2} \Rightarrow f(\xi) = A\int_\infty^\xi e^{-s^2} ds + B$

Satisfying the BC's : $\boxed{f(\xi) = \operatorname{erfc}(\xi).}$ ... erfc is the "complementary error function".

Thus, $u(y,t) = U \operatorname{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right).$