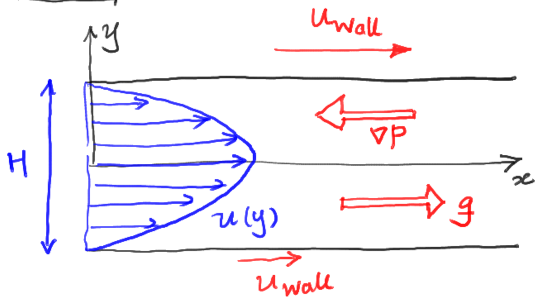
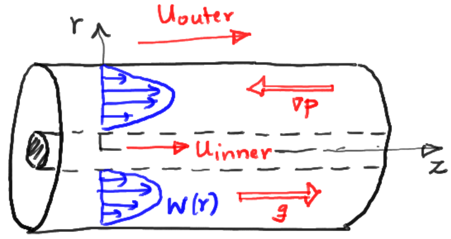
STEADY ONE-DIMENSIONAL FLOWS

Examples:



Plane Couette-Poisseuille flow



Cylindrical Couette-Poiseuille flow (also see Hagen-Poiseuille)

Circular Couette flow

One-dimensionality: Flow along one coordinate direction which varies with a perpendicular coordinate. (Note: acceleration along flow vanishes.)

Driving forces:

1. Body force g: e.g. gravity
2. Pressure gradient ∇p:
3. Motion of walls:

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p + \rho g + \mu \nabla^2 \underline{u}$$

Plane Poisseuille flow (driven by a pressure gradient).

Mass: $\cancel{\frac{\partial u}{\partial x}} + \cancel{\frac{\partial v}{\partial y}} = 0$ ... satisfied.

Momentum: $\rho\left[\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}}\right] = -\frac{\partial p}{\partial x} + \cancel{\rho g_x} + \mu\left[\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right]$

$$\underbrace{\rho\left[\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial x}} + \cancel{v\frac{\partial v}{\partial y}}\right]}_{\text{acceleration}\,=\,0} = -\frac{\partial p}{\partial y} + \cancel{\rho g_y} + \mu\left[\cancel{\frac{\partial^2 v}{\partial x^2}} + \cancel{\frac{\partial^2 v}{\partial y^2}}\right].$$

$u = u(y)$, $v = 0$, $p = ?$, $g_x = g_y = 0$.

$$\boxed{\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}}, \qquad \frac{\partial p}{\partial y} = 0 \ldots \quad p(x) \text{ alone.}$$

$$\left(\frac{dp}{dx}\right) = \mu \frac{\partial^2 u}{\partial y^2} = \cancel{X} \text{ const.} \quad \Rightarrow \quad \mu \frac{\partial u}{\partial y} = y\frac{dp}{dx} + A$$

2^{nd} order ODE for $u(y)$.

$$\mu u = \frac{y^2}{2}\frac{dp}{dx} + Ay + B \quad \text{... Solution of the ODE.}$$

Boundary conditions:

$$u\left(y = \pm\frac{H}{2}\right) = 0$$

$$\mu u = \frac{1}{2}\frac{dp}{dx}\left(y^2 - \frac{H^2}{4}\right).$$

$$\boxed{u(y) = \frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - \frac{H^2}{4}\right)}$$

u_{max} = maximum speed $= -\frac{1}{2\mu}\frac{dp}{dx}\cdot\frac{H^2}{4}$. (note $\frac{dp}{dx} < 0$).

u_{avg} = average speed $= \frac{1}{H}\int_{-H/2}^{H/2} u(y)\,dy = \frac{1}{H}\left[\frac{1}{2\mu}\frac{dp}{dx}\right]\left[\frac{y^3}{3} - y\frac{H^2}{4}\right]_{-H/2}^{H/2}$

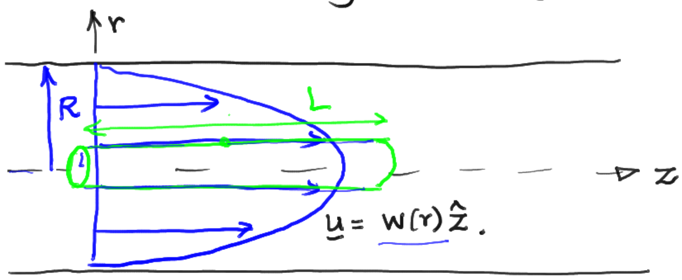
$= \frac{-1}{12\mu}\frac{dp}{dx}\cdot H^2$. $\boxed{u_{avg} = \frac{2}{3} u_{max}}$

Calculate stress & force of drag on the walls.

Hagen-Pousseuille flow (cylindrical coords drive by pressure gradient).

$\frac{\partial p}{\partial r} = 0$

$\boxed{\frac{\partial p}{\partial z}} = \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)$ = constant



$\frac{d}{dr}\left(r\frac{dw}{dr}\right) = \frac{r}{\mu}\frac{dp}{dz}$

$r\frac{dw}{dr} = \frac{r^2}{2\mu}\frac{dp}{dz} + A$. ... note

Boundary conditions

$w(r=R) = 0$

$\frac{dw}{dr} = \frac{r}{2\mu}\frac{dp}{dz} + \frac{A}{r}$

$\mu r\frac{dw}{dr} = \underbrace{\frac{r^2}{2}\frac{dp}{dz} + A\mu}$.

$w(r) = \frac{r^2}{4\mu}\frac{dp}{dz} + \boxed{A\ln r} + B$.

$F = \int_{\partial\Omega} \sigma_{rz}\,dA = \int_0^{2\pi} r\,d\theta \int_0^L dz \;\underbrace{\mu\frac{dw}{dr}} = 2\pi L\left[\frac{r^2}{2}\frac{dp}{dz} + A\mu\right]$.

$\lim_{r\to 0} F = 2\pi L\cdot A\mu. = 0$ because no physical agency is exerting this force

$\Rightarrow A = 0$.

$\boxed{w(r) = \frac{1}{4\mu}\frac{dp}{dz}\left(r^2 - R^2\right)}$.

$\boxed{w_{avg} = \frac{1}{2} w_{max}}$

$w_{max} = \frac{-1}{4\mu}\frac{dp}{dz}R^2$.

$w_{avg} = \frac{1}{\pi R^2}\int_0^R 2\pi r\,dr\cdot w(r) = -\frac{R^2}{8\mu}\frac{dp}{dz}$