Incompressible Potential flow – General considerations (§6.1 & 6.2)

1. Definition : $\underline{u} = \nabla\phi$ where ϕ = velocity potential

ϕ is a scalar.

Necessary & sufficient : $\underline{\omega} = \nabla \times \underline{u} \equiv \underline{0} \iff \underline{u} = \nabla\phi$.

(a) Mass conservation : $\nabla\cdot\underline{u} = 0 \Rightarrow \nabla^2\phi = 0$ ... Laplace eq^n

(b) Momentum conservation : $\rho\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}|\underline{u}|^2\right] + p - \rho\underline{g}\cdot\underline{x} = 0$. gives pressure.

To determine a potential flow, solve $\nabla^2\phi = 0$ with $\nabla\phi\cdot\hat{n}$ = given on boundaries.

Solution in two-dimensions – complex variables

$$\underline{\omega} = \nabla\times\underline{u} = \hat{e}_z\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \underline{0} \Rightarrow u = \frac{\partial\phi}{\partial x}, \quad v = \frac{\partial\phi}{\partial y}$$

$$\nabla\cdot\underline{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x}$$

Thus, $W(z) = \phi(x,y) + i\psi(x,y)$

where $z = x + iy$

is a holonomic function.

Cauchy-Reimann conditions
$$\left\{\begin{aligned} \frac{\partial\phi}{\partial x} &= \frac{\partial\psi}{\partial y} \\ \frac{\partial\phi}{\partial y} &= -\frac{\partial\psi}{\partial x} \end{aligned}\right.$$

Two properties of ϕ & ψ.

$\boxed{\psi = \text{stream function}}$

(i) Contours of constant ψ are streamlines.

$$\nabla\psi\cdot\underline{u} = \frac{\partial\psi}{\partial x}u + \frac{\partial\psi}{\partial y}v = (-v)u + (u)v = 0$$

(ii) Contours of constant ψ & ϕ are mutually orthogonal

$$\nabla\phi\cdot\nabla\psi = \frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y} = (u)(-v) + v(u) = 0$$