CONSERVATION OF MASS

RECAP: General conservation law for "B" with density b

$$\frac{D}{Dt}\int_{\Omega} b \, d\Omega = \int_{\Omega} q^B \, d\Omega + \int_{\partial A} T^B_{\cdots j} \, n_j \, dA \quad \ldots \text{ integral form}$$

$$\frac{\partial b}{\partial t} + \frac{\partial}{\partial x_j}(b u_j) = q^B + \frac{\partial}{\partial x_j} T^B_{\cdots j} \quad \ldots \text{ differential form}$$

q^B = volumetric source of B

$T^B_{\cdots j}$ = surface exchange rate of B on a Lagrangian boundary

In this video, B = mass.

$b = \rho$ = mass density

$q^B \equiv 0 \Leftarrow$ no volumetric generation of mass possible

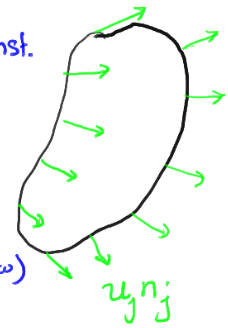
$T^B_{\cdots j} = 0 \Leftarrow$ no net exchange of mass on a Lagrangian boundary

INTEGRAL FORM

For an arbitrary Lagrangian volume Ω

Conservative Lagrangian form: $\displaystyle \frac{D}{Dt}\int_{\Omega} \rho \, d\Omega = 0 \Rightarrow M = \int_{\Omega} \rho \, d\Omega = \text{const.}$

Conservative Eulerian form: $\displaystyle \int_{\Omega} \frac{\partial \rho}{\partial t} + \int_{\partial\Omega} \rho u_j n_j \, dA = 0$



ρu_j = convective flux of mass (mass carried by the flow)

$\rho u_j n_j = \rho \underline{u} \cdot \hat{n}$ = mass crossing a unit surface with normal $\hat{n}$.

Applying divergence theorem: $\displaystyle \int_{\Omega} \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\right] d\Omega = 0$

Differential forms

Conservative : $\displaystyle \frac{\partial \rho}{\partial t} + \underbrace{\nabla \cdot (\rho \underline{u})} = 0$ OR $\displaystyle \frac{\partial \rho}{\partial t} + \underbrace{\frac{\partial (\rho u_j)}{\partial x_j}} = 0.$

rate of depletion of mass by the flow per unit volume of an infinitesimal volume

Lagrangian : $\displaystyle \frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla \cdot \underline{u} = \frac{-1}{V}\frac{DV}{Dt}$

$$V \frac{D\rho}{Dt} + \rho \frac{DV}{Dt} = 0$$

$$\boxed{\frac{D}{Dt}(\rho V) = 0}$$

Aside:

$$\underbrace{\frac{\partial \rho}{\partial t} + \underline{u} \cdot \nabla \rho} + \rho \nabla \cdot \underline{u} = 0$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \underline{u} = 0.$$

Incompressible fluid: $\boxed{\nabla \cdot \underline{u} = 0}$ OR $\displaystyle \frac{D\rho}{Dt} = 0.$ (Incompressible flow)

(constant density)