§2.4 LAGRANGIAN AND EULERIAN DESCRIPTION OF FLOW

A theory of fluid flow needs a description of the fluid motion and deformation. Two approaches are commonly used for this description, the LAGRANGIAN and the EULERIAN description.

§2.4.1 LAGRANGIAN Description (named after Joseph-Louis Lagrange)

In this description, a label is applied to infinitesimal material volumes of fluid. These material volumes themselves move with the velocity of the fluid.

A finite Lagrangian volume of fluid evolves as a collection of infinitesimal Lagrangian volumes contained within.

For mathematical purposes, the label we apply to Lagrangian fluid particles can be the position vector of the particle $\underset{\sim}{X}$ at some reference time $t=t_0$. In this way attention is focused on material fluid instead of its location.

§2.4.2 EULERIAN Description (named after Leonhard Euler):

The fluid particles are labelled by their current position. For example, if $\underset{\sim}{u}(\underset{\sim}{x},t)$ is the velocity field of the fluid continuum, it means that the fluid particle that occupies location $\underset{\sim}{x}$ at time t is moving with speed $\underset{\sim}{u}$. In this way, attention is focused on regions of space, rather than the identity of the material fluid.

§2.4.3 Eulerian versus Lagrangian descriptions.

There are two separate requirements from the flow description:

(i) Convenient representation of acceleration of material fluid particles.
LAGRANGIAN.

(ii) Convenient identification of mechanical interaction with neighbouring fluid particles.

Eulerian to Lagrangian translation

Let $\underline{F}(\underline{X},t)$ = position of Lagrangian particle labeled $\underline{X}$ at time t.

$\Rightarrow$ Lagrangian velocity field $\quad \underline{U}(\underline{X},t) = \dfrac{\partial \underline{F}}{\partial t}$.

To construct $\underline{F}$ from the Eulerian $\underline{u}(\underline{x},t)$, solve

$$\left.\frac{\partial \underline{F}}{\partial t}\right|_{X} = \underline{u}\big(\underline{F}(\underline{X},t),t\big) \qquad \text{with } \underline{F}(\underline{X},t_0) = \underline{X}.$$

To construct $\underline{u}(\underline{x},t)$ from $\underline{F}(\underline{X},t)$, write

$$\underline{u}(\underline{x},t) = \underline{U}\big(\underline{F}^{-1}(\underline{x},t),t\big),$$ where $\underline{F}^{-1}$ is the inverse Lagrangian map.

§ 2.4.4 Material derivative and acceleration

Consider any material property $c(\underline{x},t)$. The property following a Lagrangian particle labeled $\underline{X}$ is $c(\underline{F}(\underline{X},t),t)$.

$$\left.\frac{\partial c}{\partial t}\right|_{\underline{X}} = \left.\frac{\partial}{\partial t}c\big(F_i(\underline{X},t),t\big)\right|_{\underline{X}}$$

$$= \left.\frac{\partial c}{\partial t}\right|_{\underline{x}} + \left.\frac{\partial c}{\partial x_i}\right|_{t}\frac{\partial F_i}{\partial t} \quad \ldots \text{ summation implied on } i.$$

$$= \left.\frac{\partial c}{\partial t}\right|_{\underline{x}} + u_i\left.\frac{\partial c}{\partial x_i}\right|_{t} \quad \ldots \text{ because } \frac{\partial F_i}{\partial t} = \text{velocity of fluid.}$$

$$= \left.\frac{\partial c}{\partial t}\right|_{\underline{x}} + \underbrace{(\underline{u}\cdot\nabla)c}_{\text{(advection term)}} \quad \ldots \text{ in vector notation.}$$

<u>Definition</u>: Material derivative / Lagrangian derivative / Substantial derivative / Total derivative

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\underline{u}\cdot\nabla) \quad \ldots$$ gives Lagrangian rate of change for Eulerian fields without converting back & forth.

Using the material derivative:

acceleration $\quad \underline{a}(\underline{x},t) = \dfrac{D\underline{u}}{Dt} = \dfrac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u}$.

§2.4.5 Reynolds transport theorem

If b is the volumetric density of a quantity B

$$B = \int_{\Omega} b \, dV$$

For a Lagrangian volume Ω,

$$\frac{D}{Dt}\int_{\Omega(t)} b \, dV = \underbrace{\int_{\Omega(t)} \frac{\partial b}{\partial t}}_{\substack{\text{Eulerian}\\ \text{rate-of-}\\ \text{change}}} + \underbrace{\int_{\partial\Omega} b(\underline{u}\cdot\hat{n})\, dA}_{\text{advection}} .$$

$\frac{d}{dt}$