HYDROSTATICS

Hydrostatics : the study of static fluids, i.e $\underline{u} \equiv 0$

Governing equations:

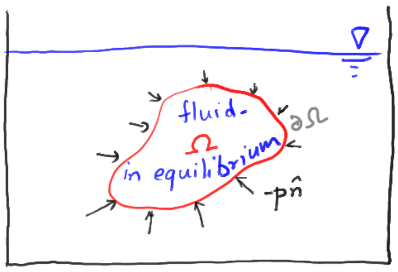
$$\left.\begin{aligned} \rho\left[\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right] &= -\nabla p + \rho \underline{g} + \mu \nabla^2 \underline{u} \\ \text{and} \quad \nabla\cdot\underline{u} &= 0 \end{aligned}\right\} \qquad \nabla p = \rho \underline{g} \quad \text{OR} \quad p = \rho \underline{g}\cdot\underline{x} + p_0 .$$

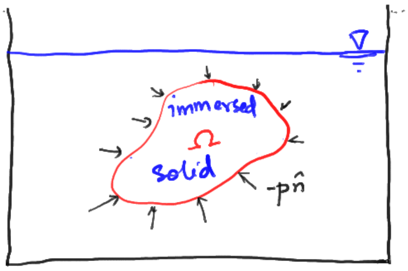
Note :

1. Fluid acceleration and viscous stress vanish
2. Pressure rises in the direction of gravity. $\underline{d}\cdot\nabla p = \rho \underline{g}\cdot\underline{d} = 0$ $\Rightarrow$ directional derivative of p along $\underline{d}$ is zero.

- Archimedes principle.

$$\underline{F} = \int_{\partial\Omega} \underline{\underline{T}}\cdot\hat{n}\, dA = \int_{\partial\Omega} -p\hat{n}\, dA = \int_{\Omega} -\nabla p\, d\Omega = -\int_{\Omega} \rho \underline{g}\, d\Omega = \text{weight of the fluid displaced by } \Omega .$$

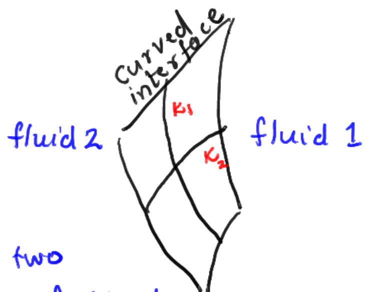




Force of buoyancy = weight of displaced fluid.

- Hydrostatics and surface tension:

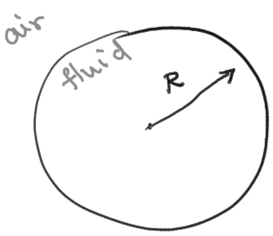
Background : Laplace law for pressure jump across an interface



$$\Delta p = \sigma(\kappa_1 + \kappa_2)$$

- $\kappa_1 + \kappa_2$ — interface curvatures along two orthogonal directions.
- σ — surface tension
- Δp — jump in pressure.

e.g.



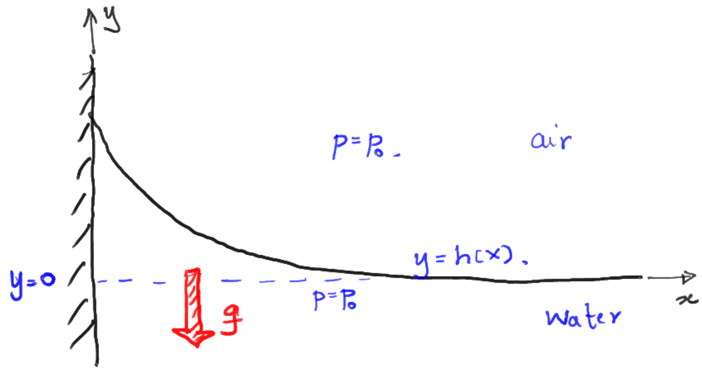
spherical drop of radius R

$$p_{\text{inside}} = p_{\text{outside}} + \frac{2\sigma}{R} .$$

$\kappa_1 = \frac{1}{R}, \quad \kappa_2 = \frac{1}{R}.$

Consequences:

(i) Shape of the air water meniscus



$$\kappa_1 = \frac{h''(x)}{\left(1+h'(x)^2\right)^{3/2}}, \qquad \kappa_2 = 0$$

$$P_{air} = P_0 = \text{constant}.$$

$$P_{water} = \underbrace{P_0}_{\text{pressure @ } y=0} - \underbrace{\rho g\, y}_{\text{hydrostatic variation of pressure w/ depth.}}$$

h/L^2 $\qquad$ h

$$\Delta p = \sigma\kappa_1 = +\rho g\, h(x) \implies \boxed{\frac{\sigma\, h''(x)}{\left[1+h'(x)^2\right]^{3/2}} = \rho g\, h(x)} \quad ★★.$$

ASIDE: A direction towards solution of this ODE. $\qquad \left[\frac{\rho g}{\sigma}\right] = \frac{1}{L^2}.$

Multiplying by $h'(x)$:

$$\text{LHS} = \frac{\sigma\, h'(x)\, h''(x)}{\left[1+h'(x)^2\right]^{3/2}} = \frac{d}{dx}\left[\frac{-\sigma}{\left[1+h'(x)^2\right]^{1/2}}\right]$$

$$\text{RHS} = \rho g\, h(x)\, h'(x) = \frac{d}{dx}\left[\tfrac{1}{2}\rho g\, h(x)^2\right].$$

$\Rightarrow$ Integrating once: $\dfrac{\sigma}{(1+h'(x)^2)^{1/2}} + \dfrac{1}{2}\rho g\, h(x)^2 = A$ ... constant of integration.

$A = \sigma$, because as $x \to \infty$, $h, h' \to 0$.

Non-dimensionalization: Note $\dfrac{\sigma}{\rho g}$ has dimensions of $((\text{length}))^2$. Define:

$$\sqrt{\frac{\sigma}{\rho g}} = \ell = \text{capillary length.} \implies \boxed{h_{max} \lesssim \sqrt{2}\,\ell}.$$

Rescale h and x by ℓ, i.e. $x = \ell\tilde{x}$ and $h = \ell\tilde{h}$.

$$★ \quad \boxed{\frac{\tilde{h}''}{(1+\tilde{h}'^2)^{3/2}} = \tilde{h}}$$

Multiply by $\tilde{h}'$ and integrate to get

$$\boxed{\frac{1}{(1+\tilde{h}'^2)^{1/2}} + \frac{\tilde{h}^2}{2} = 1.}$$

i.e. meniscus shape is universal because the dimensionless version of the governing equation has no parameters.

Separable equation, can be solved.

$$\boxed{2\sqrt{1-\frac{h^2}{4}} + \frac{1}{2}\log\left[\frac{1-\sqrt{1-h^2/4}}{1+\sqrt{1-h^2/4}}\right] = x - x_0.}$$

$\tilde{h}' \to \infty$

$\Rightarrow \tilde{h} = \sqrt{2}.$

ASIDE:

$$\left(1-\frac{h^2}{2}\right)^2 = \frac{1}{1+h'^2} \Rightarrow h'^2 = \frac{1}{\left(1-\frac{h^2}{2}\right)^2} - 1 = \frac{h^2 - h^4/4}{(1-h^2/2)^2}$$

$$\Rightarrow \quad h' = \frac{-h\,(1-h^2/4)^{1/2}}{1-h^2/2} \quad \ldots \quad \text{let} \quad 1-\frac{h^2}{4} = z \Rightarrow -\frac{h h'}{2} = z' \ \ \& \ \ h^2 = 4(1-z)$$

$$1-\frac{h^2}{2} = 1-2(1-z) = 2z-1$$

$$2z' = \frac{-4(1-z)z^{1/2}}{(2z-1)} = \frac{2(1-z)z^{1/2}}{(1/2-z)} \Rightarrow z' = \frac{(1-z)z^{1/2}}{(1/2-z)} = \frac{1/2\, z^{1/2}}{(1/2-z)} + z^{1/2}$$

$$z'\,\frac{(1-z-1/2)}{(1-z)z^{1/2}} = 1 \Rightarrow z'\left[\frac{1}{z^{1/2}} - \frac{1}{2z^{1/2}}\cdot\frac{1}{1-z}\right] = 1.$$

Let $z^{1/2} = t \Rightarrow z = t^2$ and $dz = 2t\,dt$. $\Rightarrow 2\cancel{t}\,t'\left[\frac{1}{\cancel{t}} - \frac{1}{2\cancel{t}}\,\frac{1}{1-t^2}\right] = 1$

$$t'\left[2 - \frac{1}{2(1-t)} - \frac{1}{2(1+t)}\right] = 1 \Rightarrow 2t + \frac{1}{2}\log\left(\frac{1-t}{1+t}\right) = x - x_0$$

$$2z^{1/2} + \frac{1}{2}\log\left[\frac{1-z^{1/2}}{1+z^{1/2}}\right] = x - x_0$$

$$2\sqrt{1-\frac{h^2}{4}} + \frac{1}{2}\log\left[\frac{1-\sqrt{1-h^2/4}}{1+\sqrt{1-h^2/4}}\right] = x - x_0.$$

Capillary length for water $= \sqrt{\frac{\sigma}{\rho g}} = \sqrt{\frac{72\times10^{-3}\ \mathrm{N/m}}{9800\ \mathrm{N/m^3}}} = 2.71\ \mathrm{mm}$