Elementary potential flows.

Recap: Potential flow $\underline{u} = \nabla\phi$, $\rho \frac{\partial \phi}{\partial t} + \frac{1}{2}\rho|\underline{u}|^2 + p - \rho \underline{g}\cdot\underline{x} = f(t)$,

where $\nabla^2\phi = 0$. (Laplace)

Elementary potential flows

Complex variables

- Uniform flow
- Point source
- Point vortex
- Point dipole
- Higher multipoles.

1. Uniform flow: $w(z) = U e^{-i\theta} z \longrightarrow z = x+iy$

$\phi(x,y) = U(x\cos\theta + y\sin\theta)$, $\psi(x,y) = U(y\cos\theta - x\sin\theta)$

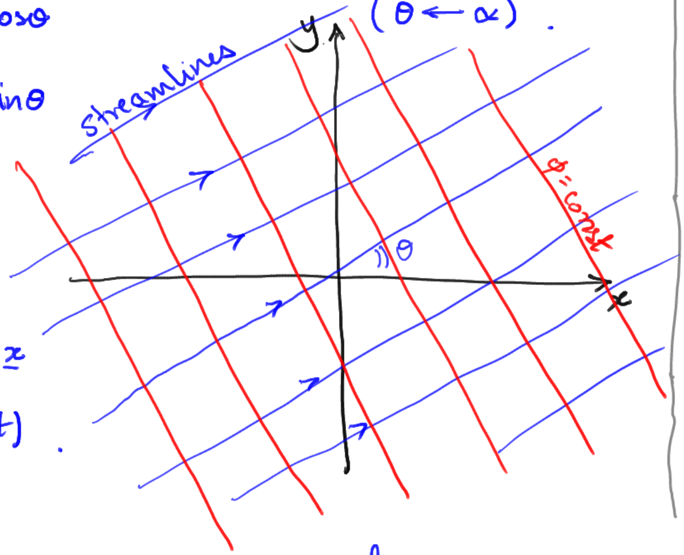
$u = \frac{\partial \phi}{\partial x} = U\cos\theta$ $\quad(\theta \leftarrow \alpha)$.

$v = \frac{\partial \phi}{\partial y} = U\sin\theta$

$\rho \frac{\partial}{\partial t}\left[U(x\cos\theta + y\sin\theta)\right]$

$+ \frac{1}{2}\rho U^2 + p - \rho \underline{g}\cdot\underline{x}$

$= f(t)$.



Aside

$e^{-i\theta} z = (\cos\theta - i\sin\theta)(x+iy)$

$= (x\cos\theta + y\sin\theta)$

$+ i\,(y\cos\theta - x\sin\theta)$

$\psi = \text{const} = c$

$U(y\cos\theta - x\sin\theta) = c$

$y = x\tan\theta + \frac{c}{U\cos\theta}$.

$\phi = \text{const} = c$

$U(x\cos\theta + y\sin\theta) = c$

$y = -\frac{x}{\tan\theta} + \frac{c}{U\sin\theta}$.

2. Point source: $w(z) = \frac{Q}{2\pi}\ln z = \frac{Q}{2\pi}\left[\ln r + i\theta\right]$ $\quad z = re^{i\theta}$.

real

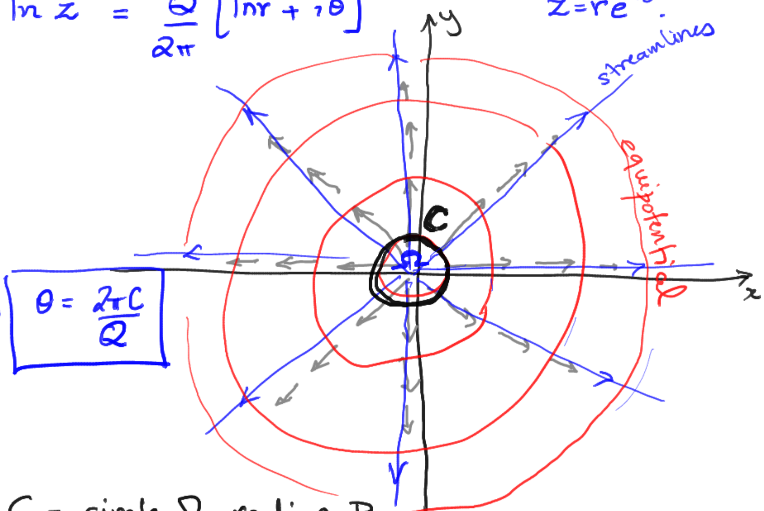
$\phi(r,\theta) = \frac{Q\ln r}{2\pi}$, $\quad \psi = \frac{Q\theta}{2\pi}$

$\underline{u} = \nabla\phi = \hat{e}_r \frac{\partial\phi}{\partial r} = \hat{e}_r \frac{Q}{2\pi r}$

Stream lines: $\psi = c \Rightarrow \frac{Q\theta}{2\pi} = c \Rightarrow \boxed{\theta = \frac{2\pi c}{Q}}$

Equipotential curves: $\phi = c$, $\frac{Q\ln r}{2\pi} = c$

$\Rightarrow \boxed{r = e^{2\pi c/Q}}$.



C = circle of radius R.

Incompressible? $\nabla\cdot\underline{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \cancel{\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}}^{0,\ u_\theta=0} = \frac{1}{r}\frac{\partial}{\partial r}\left(\cancel{r}\cdot\frac{Q}{2\pi\cancel{r}}\right) = 0.$ $r\neq 0$

flow conserves mass.

$$\int_\Omega (\nabla\cdot\underline{u})\, d\Omega = \int_{\partial\Omega} \underline{u}\cdot\hat{n}\, dA$$

$\text{RHS} = \int_0^{2\pi} \hat{e}_r \frac{Q}{2\pi\cancel{r}}\cdot\hat{e}_r\cdot\cancel{r}\,d\theta = Q.$ $r=R\neq 0$

$$\boxed{\nabla\cdot\underline{u} = Q\,\delta(r).}$$

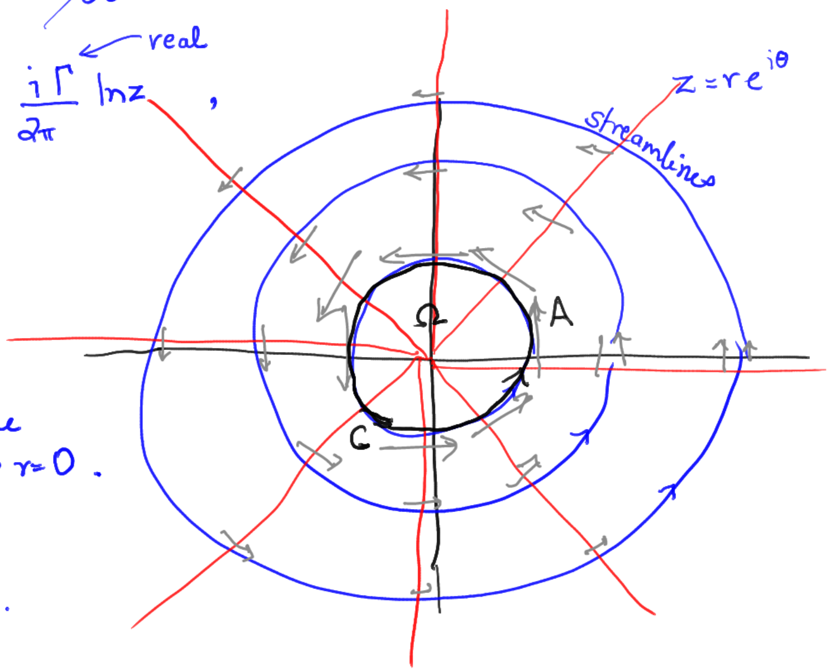
hence "point source".

$\text{LHS} = ?$ $\quad \nabla\cdot\underline{u} = 0 \quad r\neq 0.$

$\underline{\omega} = \nabla\times\underline{u} = \hat{e}_z\frac{1}{r}\left[\cancel{\frac{\partial}{\partial r}(r u_\theta)}^{0,\ u_\theta=0} - \cancel{\frac{\partial u_r}{\partial\theta}}^{u_r(r)}\right] = \underline{0}$

Q = strength of the point source.

3. Point vortex: $\quad W(z) = \frac{i\Gamma}{2\pi}\ln z$, real



$\phi = \frac{\Gamma\theta}{2\pi}, \quad \psi = -\frac{\Gamma}{2\pi}\ln r$

$\underline{u} = \nabla\phi = \frac{\hat{e}_\theta}{r}\frac{\partial\phi}{\partial\theta} = \frac{\hat{e}_\theta\Gamma}{2\pi r}.$

$\nabla\cdot\underline{u} = \cancel{\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}} = 0$ everywhere except @ $r=0$.

$$\int_\Omega \nabla\cdot\underline{u}\, d\Omega = \int_{\partial\Omega}\underline{u}\cdot\hat{n}\, dA \equiv 0.$$

$\underline{u}\cdot\hat{n} = \frac{\hat{e}_\theta\Gamma}{2\pi r}\cdot\hat{e}_r = 0$

$\Rightarrow \quad \nabla\cdot\underline{u} \equiv 0$ for this flow.

$\nabla\times\underline{u} = \hat{e}_z\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) = \frac{\hat{e}_z}{r}\cancel{\frac{\partial}{\partial r}\left(\frac{\Gamma}{2\pi}\right)}^{0} = \underline{0}\ \ldots \quad r>0$

$$\int_A \nabla\times\underline{u}\, dA = \boxed{\int_{\partial A}\underline{u}\cdot\hat{t}\, ds} \ \ldots \text{ Stokes theorem.}$$

$\left(\text{Eq } 6{\cdot}22 \ \int_C \underline{u}\cdot\hat{t}\, ds\right)$

$\nabla\times\underline{u} = \Gamma\cdot\delta(r)\,\hat{e}_z.$

$\text{RHS} = \int_0^{2\pi}\hat{e}_\theta\frac{\Gamma}{2\pi\cancel{r}}\cdot\hat{e}_\theta\cdot\cancel{r}\,d\theta$ $r=R\neq 0$

$= \Gamma.$

$\boxed{\Gamma = \text{circulation of the point vortex}}$

Circulation about C $\equiv \int_C \underline{u}\cdot\hat{t}\, ds.$

$= \int (\nabla\times\underline{u})\cdot\hat{e}_z\, dA$ (by Stokes theorem)

Aside:

$\oint_C \underline{u}\cdot\underline{t}\, ds = \Gamma$ if $z=0$ is inside C

$= 0$ otherwise.