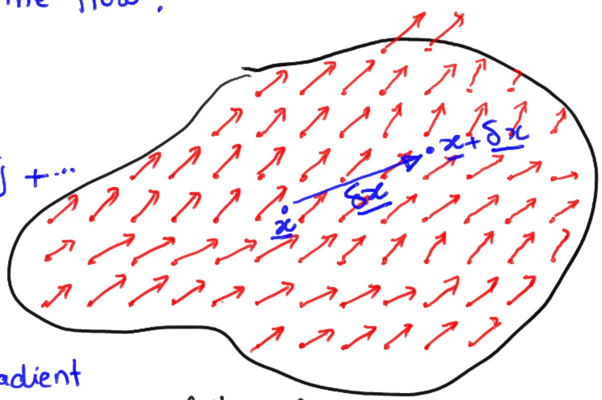
DEFORMATION OF INFINITESIMAL FLUID ELEMENTS

- "Continuous deformation" in the definition of a fluid
- Deformation of finite volumes $\longleftrightarrow$ collection of infinitesimal deforming volumes

In the most general case, consider where the neighbouring points in an infinitesimal volume are transported by the flow.



Infinitesimal volume

Few "modes" of deformation:

$$u_i(x_j+\delta x_j,t) = u_i(x_j,t) + \frac{\partial u_i}{\partial x_j}(x_j,t)\,\delta x_j + \ldots$$

$$u_i(x_j+\delta x_j,t) = u_i(x_j,t) + r_{ij}\,\delta x_j + e_{ij}\,\delta x_j$$

where $r_{ij} + e_{ij} = \frac{\partial u_i}{\partial x_j}$ ... the velocity gradient tensor.

$$r_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \text{anti-symmetric part of } \frac{\partial u_i}{\partial x_j}. \quad \text{OR} \quad \underline{\underline{r}} = \frac{1}{2}\left(\underline{\underline{\nabla u}} - \underline{\underline{\nabla u}}^T\right)$$

$r_{ij} = -r_{ji}$.

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \text{symmetric part of } \frac{\partial u_i}{\partial x_j} \quad \text{or} \quad \underline{\underline{e}} = \frac{1}{2}\left(\underline{\underline{\nabla u}} + \underline{\underline{\nabla u}}^T\right).$$

$\underline{\underline{r}}$ = rate of rotation-tensor.

$\underline{\Omega} = \frac{1}{2}(\nabla \times \underline{u}) = \frac{1}{2}\underline{\omega}$, where

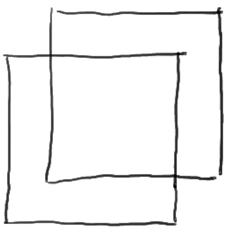
$\underline{\omega} = \nabla \times \underline{u}$ = vorticity.

$r_{ij} = \varepsilon_{ijk}\Omega_k \longleftrightarrow$ anti-symmetry in (i,j)

$$\Rightarrow r_{ij}\,\delta x_j = \varepsilon_{ijk}\,\delta x_j\,\Omega_k = \underbrace{(\underline{\delta x} \times \underline{\Omega})_i}_{\text{rigid body rotation velocity}}$$

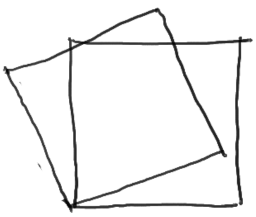
$$\underline{u}(\underline{x}+\underline{\delta x},t) = \underbrace{\underline{u}(\underline{x},t)}_{\text{rigid translation}} + \underbrace{\underline{\underline{r}}\cdot\underline{\delta x}}_{\text{rigid rotation}} + \underbrace{\underline{\underline{e}}\cdot\underline{\delta x}}_{\text{deformation}}$$

$\boxed{\underline{\underline{e}} = \text{rate of deformation tensor}}$



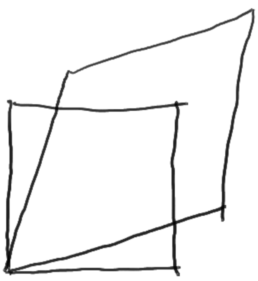
Translation

$\underline{u}(\underline{x},t)$.



Rotation

$\underline{\underline{r}}$



Deformation

$\underline{\underline{e}}$

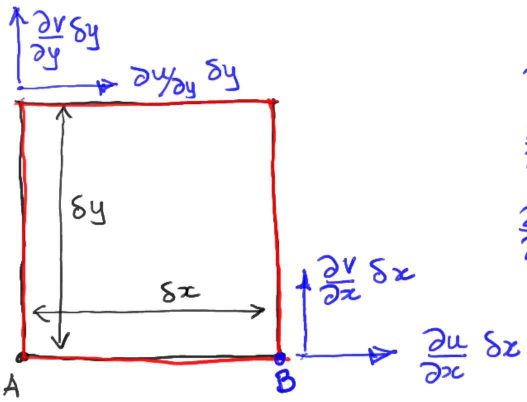
$$\underline{\underline{e}} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ & e_{22} & e_{23} \\ \text{(symmetric)} & & e_{33} \end{bmatrix} \quad \ldots \text{ six independent modes of deformation}$$

(symmetric tensor)

Extensional rates of deformation : e_{11}, e_{22}, e_{33} (diagonal elements)

Shear rates of deformation : e_{12}, e_{13}, e_{23}. (off-diagonal elements)

e.g. say $\underline{u} = (u, v, w)$ (but we will draw a 2D picture)



$\frac{\partial u}{\partial x} = e_{11}$ = extensional strain rate along x

$\frac{\partial v}{\partial y} = e_{22}$ = —————— " —————— y

$\frac{\partial w}{\partial z} = e_{33}$ = —————— " —————— z

Volume $\delta V = \delta x\, \delta y\, \delta z$

$$\frac{1}{\delta V}\frac{d\delta V}{\delta t} = \frac{1}{\delta x}\frac{d\delta x}{dt} + \frac{1}{\delta y}\frac{d\delta y}{\delta t} + \frac{1}{\delta z}\frac{d\delta z}{dt}$$

$$= e_{11} + e_{22} + e_{33}$$

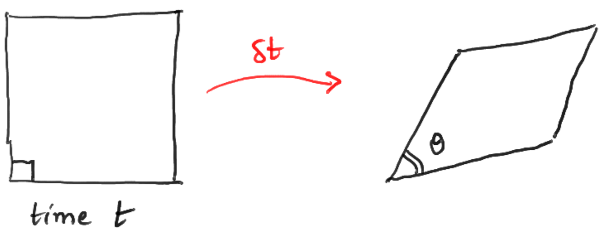
$$= \text{trace}(\underline{\underline{e}}) = \nabla\cdot\underline{u}.$$

$\frac{1}{\delta x}\frac{d\delta x}{dt} = \frac{\partial u}{\partial x}$, and so on.

$e_{12} = e_{21} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$ = shear rate in the xy-plane.

$e_{13} = e_{31} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$ = ———— " —— xz-plane

$e_{23} = e_{32} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$ = ———— " —— yz-plane.



$-\frac{\partial \theta}{\partial t} = 2e_{12} \ldots$ (Examples sheet 2).

and similarly for e_{13}, e_{23}.

<u>Two examples</u>:

Extensional flow

$(u, v) = (x, -y)$

$$\underline{\underline{e}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

shear flow

$(u, v) = (y, x)$

$$\underline{\underline{e}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$\underline{\underline{e}}$ = deformation rate tensor $\longleftrightarrow$ symmetric.

$$\hat{e}_1 = a_{11}\hat{v}_1 + a_{12}\hat{v}_2 + a_{13}\hat{v}_3$$

$$\hat{e}_2 = a_{21}\hat{v}_1 + a_{22}\hat{v}_2 + a_{23}\hat{v}_3$$

$$\hat{e}_3 = a_{31}\hat{v}_1 + a_{32}\hat{v}_2 + a_{33}\hat{v}_3 .$$

$$\underline{\underline{e}} \cdot \underline{v} = \lambda \underline{v}$$

$$\left.\begin{array}{l} \lambda_1 , \underline{v}_1 \\ \lambda_2 , \underline{v}_2 \\ \lambda_3 , \underline{v}_3 \end{array}\right\} \text{eigenvectors are orthogonal}$$

$$\underline{\underline{e}} \xrightarrow[\text{system}]{\text{new coordinate}} \underline{\underline{e}}' = \begin{bmatrix} e'_{11} & 0 & 0 \\ 0 & e'_{22} & 0 \\ 0 & 0 & e'_{33} \end{bmatrix}$$

$$\left.\begin{array}{l} e'_{11} = \lambda_1 \\ e'_{22} = \lambda_2 \\ e'_{33} = \lambda_3 . \end{array}\right\} \text{principal values of } \underline{\underline{e}} .$$