CONSTITUTIVE LAWS

Background + Recap

Mass conservation : $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0$ OR (incompressible) $\nabla \cdot \underline{u} = 0$

Momentum conservation : $\rho\left[\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u}\right] = \rho g + \nabla \cdot \underline{\underline{T}}$

Angular momentum : $\underline{\underline{T}}^T = \underline{\underline{T}}$.

Deformation rate tensor $\underline{\underline{e}} = \frac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T)$.

PROBLEM: NEED TO KNOW $\underline{\underline{T}}$. Enter Constitutive law.

Constitutive law: Express $\underline{\underline{T}}$ in terms of deformation rate tensor.

$$T_{ij} = f[e_{mn}; p] \ldots$$ most general form (given $T_{ij} = T_{ji}$).

An ideal fluid - the simplest constitutive law

$$T_{ij} = -p\,\delta_{ij}$$ where p is the fluid pressure.

$$\rho\left[\frac{\partial u}{\partial t} + \underline{u} \cdot \nabla \underline{u}\right] = \rho g - \nabla p .$$

OR $$\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} = g - \frac{1}{\rho}\nabla p$$

Euler equation for an ideal fluid.

$$\frac{\partial T_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_j}\delta_{ij} = -\frac{\partial p}{\partial x_i}$$

$$\nabla \cdot \underline{\underline{T}} = -\nabla p$$

How to determine p?

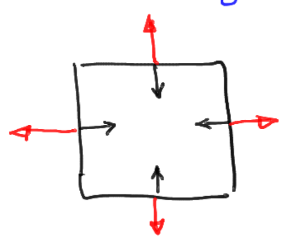
- Compressible flows:
  - Pressure changes with density only, i.e. $p(\rho)$, which closes the system of equations. Examples:

    Iso-thermal gas : $p = \rho RT$, $T =$ constant

    Isentropic gas : $p = c\rho^n$.
  - Pressure changes with density and temperature, i.e $p(\rho, T)$. Use conservation of energy to derive an equation for temperature T.

- Incompressible flows: $\nabla \cdot \underline{u} = 0$ — Determine p self-consistently

$$\underline{u}^{\text{future}} = \underline{u}^{\text{now}} + \delta t\left[-\underline{u} \cdot \nabla \underline{u} + g - \frac{1}{\rho}\nabla p\right].$$



Pick a p such that $\nabla \cdot \underline{u}^{\text{future}} = 0$.

An ideal fluid is an inviscid fluid.

A NEWTONIAN FLUID

$$T_{ij} = -p\,\delta_{ij} + \sigma_{ij} \qquad \text{where} \quad \sigma_{ij} = S_{ij}(e_{mn};\rho)$$

Total stress — Deviatoric OR viscous stress

– Linearity, i.e. $\sigma_{ij} = \underbrace{A_{ijmn}}_{\text{constant fourth rank tensor}} \underline{\underline{e_{mn}}}$.

– Isotropy, i.e. components A_{ijmn} do not depend on coordinate rotation.

$$A_{ijmn} = \lambda\,\delta_{ij}\delta_{mn} + \alpha\,\delta_{im}\delta_{jn} + \beta\,\delta_{in}\delta_{jm}.$$

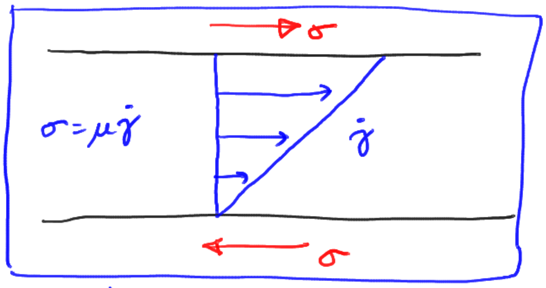
Aside: $\alpha\,\underbrace{\delta_{im}\delta_{jn}}\,\underbrace{e_{mn}} = \alpha\,\delta_{im}e_{mj} = \alpha\,\delta_{ij}$

$$\Rightarrow \quad \underline{\underline{\sigma_{ij}}} = \lambda\,\delta_{ij}\,\cancelto{0}{e_{mm}} + (\alpha+\beta)\,e_{ij}.$$

(We take $e_{mm} = \nabla\cdot\underline{u} = 0$ from now on.)

Determine coefficients by comparing to the Newtonian case.

$$\boxed{\sigma_{ij} = 2\mu\,e_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}$$



Incompressible fluid.

Then, $$\frac{\partial T_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(-p\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right)$$

$$= -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} \qquad \text{or} \qquad \nabla\cdot\underline{\underline{T}} = -\nabla p + \mu\nabla^2\underline{u}.$$

Conservation of momentum:

$$\underbrace{\rho\left[\frac{\partial \underline{u}}{\partial t} + \underbrace{(\underline{u}\cdot\nabla)\underline{u}}_{\text{advection term}}\right]}_{\text{inertial term}} = \underbrace{-\nabla p}_{\text{pressure gradient}} + \underbrace{\rho \underline{g}}_{\text{body force}} + \underbrace{\mu\nabla^2\underline{u}}_{\text{viscous stress}} \ldots$$

(pause & write in index form).

+ $\nabla\cdot\underline{u} = 0$ gives the Navier-Stokes equation.

☆ Comment about cyclindrical coordinates. (Wikipedia)