CONSERVATION OF MOMENTUM

RECAP:

$$\frac{D}{Dt}\int_\Omega b\, d\Omega = \int_\Omega q^B\, d\Omega + \int_{\partial\Omega} T^B_{\dots j}\, n_j\, dA$$

... integral form of conservation law for B.

This video: B = momentum (rank 1 tensor i.e. a vector).

$b = \rho \underline{u}$ (density of momentum)

$q^B = \rho \underline{g}$ volumetric force density, a.k.a body force

Rate of change of momentum is a force, so the generation rates are all forces.

e.g. the force of gravity, where $\underline{g}$ = acceleration due to gravity.

$T_{ij} = T^B_{ij}$ = Cauchy stress tensor, a second rank tensor that can be used to represent forces transmitted across surfaces.

Conservation of momentum in integral form

$$\frac{D}{Dt}\int_\Omega \rho \underline{u}\, d\Omega = \int_\Omega \rho \underline{g}\, d\Omega + \int_{\partial\Omega} \underline{\underline{T}}\cdot\hat{n}\, dA \quad \text{OR} \quad \frac{D}{Dt}\int_\Omega \rho u_i\, d\Omega = \int_\Omega \rho g_i\, d\Omega + \int_{\partial\Omega} T_{ij}\, n_j\, dA.$$

Using Reynolds transport theorem for the L.H.S.

$$\int_\Omega \frac{\partial(\rho\underline{u})}{\partial t} d\Omega + \int_{\partial\Omega} \rho\underline{u}(\underline{u}\cdot\hat{n})\, dA = \int_\Omega \rho\underline{g}\, d\Omega + \int_{\partial\Omega} \underline{\underline{T}}\cdot\hat{n}\, dA$$

$$\text{OR} \quad \int_\Omega \frac{\partial}{\partial t}(\rho u_i)\, d\Omega + \int_{\partial\Omega} \rho u_i (u_j n_j)\, dA = \int_\Omega \rho g_i\, d\Omega + \int_{\partial\Omega} T_{ij} n_j\, dA.$$

Using divergence theorem

$$\int_\Omega \left[\frac{\partial}{\partial t}(\rho\underline{u}) + \nabla\cdot(\rho\underline{u}\,\underline{u}) - \rho\underline{g} - \nabla\cdot\underline{\underline{T}}\right] d\Omega = \underline{0} \quad \text{OR} \quad \int_\Omega \left[\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j) - \rho g_i - \frac{\partial T_{ij}}{\partial x_j}\right] d\Omega = 0$$

Conservation of momentum in differential form

$$\frac{\partial}{\partial t}(\rho\underline{u}) + \nabla\cdot(\rho\underline{u}\,\underline{u}) = \rho\underline{g} + \nabla\cdot\underline{\underline{T}} \quad \text{OR} \quad \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \rho g_i - \frac{\partial T_{ij}}{\partial x_j}.$$

Note that

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \underbrace{\rho\left[\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right]}_{\text{acceleration } \underline{a}} + u_i \underbrace{\cancel{\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\right]}}_{0,\ \text{mass conservation}}$$

$$\Rightarrow \quad \rho\frac{D\underline{u}}{Dt} = \rho\underline{g} + \underbrace{\nabla\cdot\underline{\underline{T}}}_{\substack{\text{surface force}\\ \text{per unit volume}\\ \text{on an infinitesimal volume.}}} \quad \text{OR} \quad \rho\frac{Du_i}{Dt} = \rho g_i + \frac{\partial T_{ij}}{\partial x_j}$$