CONSERVATION LAWS IN CONTINUUM MECHANICS.

- Rate of change = Sources − sinks.

Basics

Quantity	Density
Mass	ρ
Momentum	$\rho \underline{u}$
Angular momentum	$\underline{x} \times \rho \underline{u}$.
Abstraction B	b

Total amount of B in Ω.

$$\boxed{B = \int_\Omega b \, d\Omega .}$$

Volumetric source $q^B(\underline{x},t)$

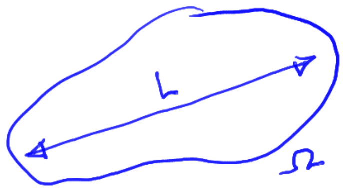
Surface source $h^B(\underline{x},t;\hat{n})$

$\partial\Omega$ = boundary of Ω.

$$\underbrace{\frac{D}{Dt}\int_\Omega b \, d\Omega}_{L^3} = \underbrace{\int_\Omega q^B d\Omega}_{L^3} + \underbrace{\int_{\partial\Omega} h^B dA}_{L^2} .$$

e.g Sources of momentum

$$\int_\Omega \rho g \, d\Omega + \int_{\partial\Omega} -p\hat{n} \, dA$$



$$\lim_{L\to 0} \frac{1}{L^2}\left\{ \frac{D}{Dt}\int_\Omega b\,d\Omega - \int_\Omega q^B d\Omega - \int_{\partial\Omega} h^B dA \right\} \Rightarrow \boxed{\int_{\partial\Omega} h^B dA = 0}$$

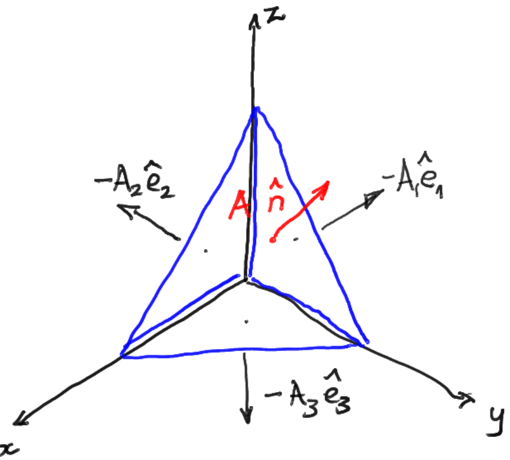
Augustine-Louis Cauchy and his tetrahedron

$\underline{\underline{A\hat{n}}} = \underline{\underline{A_i \hat{e}_i}}$ OR $\boxed{An_i = A_i}$ ←

$h^B(\hat{n})A + h^B(-\hat{e}_1)A_1 + h^B(-\hat{e}_2)A_2 + h^B(-\hat{e}_3)A_3 = 0.$ ←

$h^B(-\hat{e}_j) = -h^B(\hat{e}_j).$

$h^B(\hat{n}) = h^B(\hat{e}_j)n_j$! ←



$$T^B_{\cdot\cdot j} = h^B(\hat{e}_j)$$

$$\boxed{h^B(\hat{n}) = T^B_{\cdot\cdot j}\, n_j}$$

Curing the dimensional inconsistency

$$\frac{D}{Dt}\int_{\Omega} b \, d\Omega = \int_{\Omega} q^B \, d\Omega + \int_{\partial\Omega} T^B_{\cdots j} \, n_j \, dA$$

Applying divergence theorem to the surface source term

$$\underbrace{\frac{D}{Dt}\int_{\Omega} b \, d\Omega}_{L^3} = \underbrace{\int_{\Omega} q^B \, d\Omega}_{L^3} + \underbrace{\int_{\Omega} \frac{\partial}{\partial x_j} T^B_{\cdots j} \, d\Omega}_{L^3}.$$

Integral forms expressing conservation law

1. Applying Reynolds transport theorem

$$\int_{\Omega} \frac{\partial b}{\partial t} d\Omega + \int_{\partial\Omega} b u_j n_j \, dA = \int_{\Omega} q^B d\Omega + \int_{\partial\Omega} T^B_{\cdots j} n_j \, dA.$$

Called the "Conservative form". Flux $= \left(b u_j - T^B_{\cdots j}\right)$.

usually diffusive flux.

convective flux or advective flux.

- Whether to apply divergence theorem
- Whether to apply Reynolds transport theorem.

2. Applying both.

$$\int_{\Omega} \left(\frac{\partial b}{\partial t} + \frac{\partial}{\partial x_j}(b u_j) - q^B - \frac{\partial}{\partial x_j} T^B_{\cdots j} \right) d\Omega = 0 \ldots \quad \text{for any } \Omega!$$

Differential form expressing conservation law

$$\boxed{\frac{\partial b}{\partial t} + \frac{\partial}{\partial x_j}(b u_j) = q^B + \frac{\partial T^B_{\cdots j}}{\partial x_j}} \quad \ldots\ldots \quad \text{Eq. } 2.25$$

Milestone : Applied conservation law to an arbitrary volume!