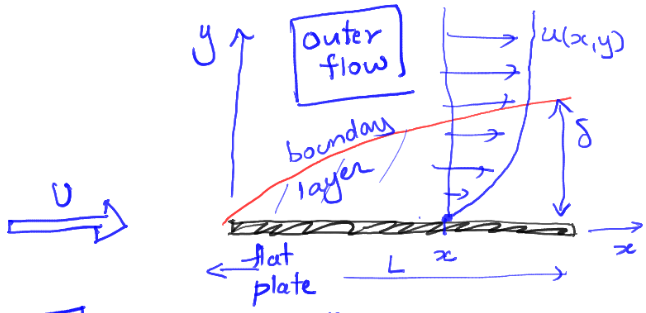
Blasius similarity solution for flow past flat plate

Recap: Prandtl boundary layer eq^{n}s

$$\rho\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right]=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2}$$

$$0=-\frac{\partial p}{\partial y}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,.$$

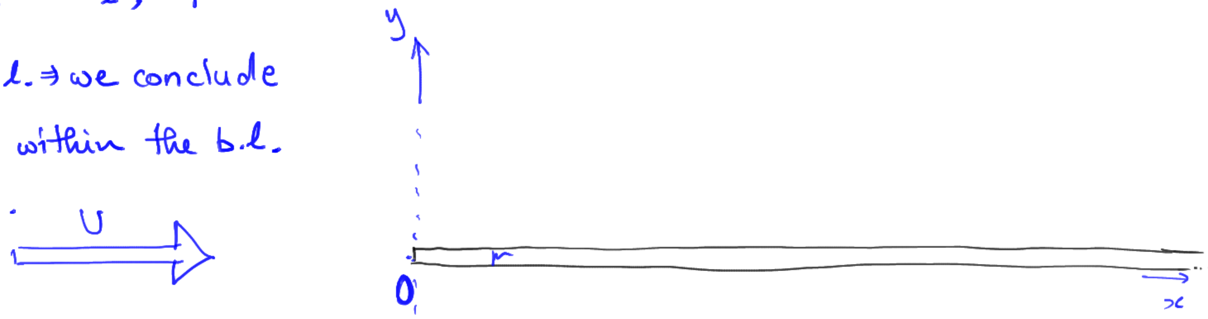


$\delta\approx\sqrt{\frac{\nu L}{U}}$ or $\delta\approx\sqrt{\frac{\nu x}{U}}$. $\delta \ll L$

Outer flow: $\underline{u}=U\hat{e}_x$, $p=$ constant.

$\frac{\partial p}{\partial y}=0$ in the b.l. $\Rightarrow$ we conclude $p=$ constant within the b.l.

$\Rightarrow \frac{\partial p}{\partial x}=0$ in the b.l.



$\nu=\frac{\mu}{\rho}$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,.$$

$u\to\alpha\tilde{u}$, $v\to\beta\tilde{v}$

$$\tilde{u}\frac{\partial\tilde{u}}{\partial\tilde{x}}+\frac{\gamma\beta}{\alpha\delta}\tilde{v}\frac{\partial\tilde{u}}{\partial\tilde{y}}=\frac{\nu\gamma}{\alpha\delta^2}\frac{\partial^2\tilde{u}}{\partial\tilde{y}^2}$$

$$\frac{\partial\tilde{u}}{\partial\tilde{x}}+\frac{\gamma\beta}{\alpha\delta}\frac{\partial\tilde{v}}{\partial\tilde{y}}=0\,.$$

@ $x=0$, $u=U$. } uniform
@ $y\to\infty$, $u=U$. } flow.

@ $y=0$, $u=0$. no slip.
$v=0$ no penetration

$x\to\gamma\tilde{x}$, $y\to\delta\tilde{y}$.

@ $\tilde{x}=0$, $\tilde{u}=U/\alpha$
@ $\tilde{y}\to\infty$, $\tilde{u}=U/\alpha$

@ $\tilde{y}=0$, $\tilde{u}=0$.
$\tilde{v}=0$

$\longrightarrow \frac{\gamma\beta}{\alpha\delta}=1$, $\frac{\gamma}{\alpha\delta^2}=1$

$\alpha=1$.

$\beta=\frac{\delta}{\gamma}$ and $\delta=\sqrt{\gamma}$

$=1/\sqrt{\gamma}$.

$u=f(x,y)$ $\qquad$ $\tilde{u}=f(\tilde{x},\tilde{y})$

$v=g(x,y)$. $\qquad$ $\tilde{v}=g(\tilde{x},\tilde{y})$.

$$\frac{u}{\alpha}=f\left(\frac{x}{\gamma},\frac{y}{\delta}\right)\quad\text{and}\quad\frac{v}{\beta}=g\left(\frac{x}{\gamma},\frac{y}{\delta}\right)$$

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$$\boxed{u=f\left(\frac{x}{\gamma},\frac{y}{\sqrt{\gamma}}\right)\quad\text{and}\quad v=\frac{1}{\sqrt{\gamma}}\,g\left(\frac{x}{\gamma},\frac{y}{\sqrt{\gamma}}\right).}$$

for an $\gamma>0$.

We pick, $\gamma=x$. $\Rightarrow$ $\boxed{u=f\left(1,\frac{y}{\sqrt{x}}\right),\quad v=\frac{1}{\sqrt{x}}\,g\left(1,\frac{y}{\sqrt{x}}\right)}$.

similarity variable is of the form $\frac{y}{\sqrt{x}}$.

Onwards to the similarity solution. (back of the envelope).

$$\longrightarrow u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{U^2}{x} = \frac{VU}{\delta} \qquad \frac{\nu U}{\delta^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

$$\frac{U}{x} = \frac{V}{\delta} \Rightarrow V = \frac{U}{x}\delta.$$

$u \sim U$, $v \sim V$

$x \sim x$, $y \sim \delta$.

$$\frac{U^2}{x} = \frac{\nu U}{\delta^2} \quad = \quad \underline{\underline{\delta = \sqrt{\frac{2\nu x}{U}}}}.$$

the factor of 2 is for historical reasons.

$$\underline{\underline{\xi = \frac{y}{\delta}}} \quad \ldots \text{ similarity variable}$$

Self-similar ansatz :

$$u = U F'(\xi)$$

$$v = ?$$

$$\xi = \frac{y}{\delta}, \qquad \delta = \sqrt{\frac{2\nu x}{U}}.$$

$$\frac{\partial \xi}{\partial x} = -\frac{\xi}{2x}, \qquad \frac{d\delta}{dx} = \frac{\delta}{2x}.$$

$$\frac{\partial \xi}{\partial y} = \frac{1}{\delta}.$$

mass conservation :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \Rightarrow \frac{1}{\delta}\frac{\partial v}{\partial \xi} = -\frac{\partial u}{\partial x} = -U\,F''(\xi).\frac{\partial \xi}{\partial x} = +\frac{U\xi}{2x}F''(\xi).$$

$$\frac{\partial v}{\partial \xi} = \sqrt{\frac{2\nu x}{U}}\cdot\frac{U}{2x}\cdot\underbrace{\xi F''(\xi)}_{\text{func of }\xi} = \sqrt{\frac{\nu U}{2x}}\,\xi F''(\xi).$$

$$\boxed{v = \sqrt{\frac{\nu U}{2x}}\left[\xi F'(\xi) - F(\xi)\right]}. \quad \text{ansatz for } v.$$

x-momentum :

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}.$$

$$UF'(\xi)\left[-\frac{\xi}{2x}.UF''(\xi)\right] + \sqrt{\frac{\nu U}{2x}}\left[\xi F' - F\right]\frac{UF''(\xi)}{\sqrt{\frac{2\nu x}{U}}} = \frac{\nu U F'''(\xi)}{\left(\frac{2\nu x}{U}\right)}.$$

$$\frac{U^2}{2x}\left[-\cancel{\xi F'F''} + [\cancel{\xi F'} - F)F''\right] = \frac{U^2}{2x}F'''(\xi).$$

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$$\boxed{F''' + FF'' = 0} \quad \text{ODE for } F.$$

w/ b.cs.

$UF'(\infty) = U$ $\quad x=0$ or $y\to\infty$. $\Rightarrow$ $F'(\infty) = 1$.

$UF'(0) = 0$ (no slip) @ $y=0$ $\Rightarrow$ $F'(0) = 0$.

$\sqrt{\frac{\nu U}{2x}}F(0) = 0$ @ $y=0, x>0$. $\Rightarrow$ $F(0) = 0$.

three boundary conditions.

$$\frac{\delta}{L} \ll 1 \quad \Rightarrow \frac{1}{L}\sqrt{\frac{\nu x}{U}} \ll 1. \quad \Rightarrow \quad \frac{\nu x}{U} \ll L^2$$

$$\boxed{\frac{x}{L} \ll \boxed{\frac{UL}{\nu}}.} \qquad \text{If } Re \gg 1. \qquad \frac{UL}{\nu} = \frac{UL}{\mu/\rho} = \frac{\rho U L}{\mu} = Re.$$

Drag on a flat plate:

$$\underline{F} = \int_{\partial\Omega} \underline{\underline{T}} \cdot \hat{n}\, dA = \underset{1.}{\cancel{W}} \int_0^L dx \begin{bmatrix} -p + 2\mu \frac{\partial u}{\partial x} & \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & -p + 2\mu \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \int_0^L dx \begin{pmatrix} \mu\left(\frac{\partial u}{\partial y} + \partial v/\partial x\right) \\ -p + 2\mu \frac{\partial v}{\partial y} \end{pmatrix}.$$

$$F_1 = \int_0^L dx \cdot \mu\left(\frac{\partial u}{\partial y} + \cancel{\frac{\partial v}{\partial x}}\right)$$

neglected $\delta/x \ll 1$.

$$= \int_0^L dx\, \mu \frac{U}{\delta(x)} F''(\xi)\Big|_{\xi=0}$$

$$= F''(0)\, \mu U. \int_0^L \frac{dx}{\sqrt{\frac{2\nu x}{U}}}.$$

$$= F''(0) \cdot \sqrt{2\mu\rho U^3 L}.$$

$$\boxed{F''(0) = 0.4696.}$$

$$\boxed{F_x \approx 0.664\sqrt{\mu\rho U^3 L}}.$$

$$\frac{\partial u}{\partial y} \sim \frac{U}{\delta}$$

$$\frac{\partial v}{\partial x} \sim \frac{V}{x} = \frac{U\delta}{x} \cdot \frac{1}{x}$$

$$\frac{U}{\delta} \gg \frac{U\delta}{x^2}.$$