APPLICATIONS OF BERNOULLI EQUATION

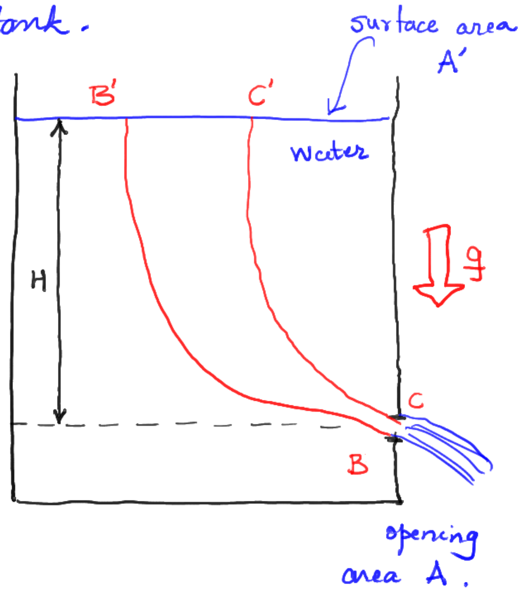
- Steady inviscid flow

$$\hat{B} = \frac{1}{2}\rho|\underline{u}|^2 + p - \rho\underline{g}\cdot\underline{x} \quad = \text{constant along a streamline}$$

- Potential flow $\underline{u} = \nabla\phi$ for some scalar ϕ

$$\tilde{B} = \rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho|\underline{u}|^2 + p - \rho\underline{g}\cdot\underline{x} = f(t) \quad \ldots \text{ a function of } t \text{ alone.}$$

Application: Flow through an opening in a tank.



Apply steady inviscid version (why?)

$$\cancel{p_B} + \frac{1}{2}\rho U_B^2 - \rho\underline{g}\cdot\underline{x}_B = \cancel{p_{B'}} + \frac{1}{2}\rho \cancel{U_{B'}^2} - \rho\underline{g}\cdot\underline{x}_{B'}.$$

① ① neglect ②

① $p_B \simeq p_{B'} = p_{atm}$.

② $U_{B'}A' = U_B A \Rightarrow U_{B'} = \left(\frac{A}{A'}\right) U_B . \ll U_B.$

(small)

$$\frac{1}{2}\rho U_B^2 = \rho\underline{g}\cdot(\underline{x}_B - \underline{x}_{B'}) = \rho g H.$$

$$\boxed{U_B = \sqrt{2gH}}$$

if $H = 0.05\,m \Rightarrow U_B = \sqrt{\underset{2}{2} \times \underset{g}{10} \times \underset{H}{0.05}} = 1\,m/s.$

Steady: $\frac{dH}{dt} \ll U_B$. ✓ $\qquad \frac{dH}{dt} = U_{B'} = \frac{A}{A'}U_B \ll U_B$. from ②.

Inviscid: Reynolds number $Re = \frac{\rho U L}{\mu} = \frac{10^3\,kg/m^3 \times 1\,m/s \times 10^{-2}\,m}{10^{-3}\,kg/ms} \simeq 10^4$

Because Re is so large, viscosity may be ignored.

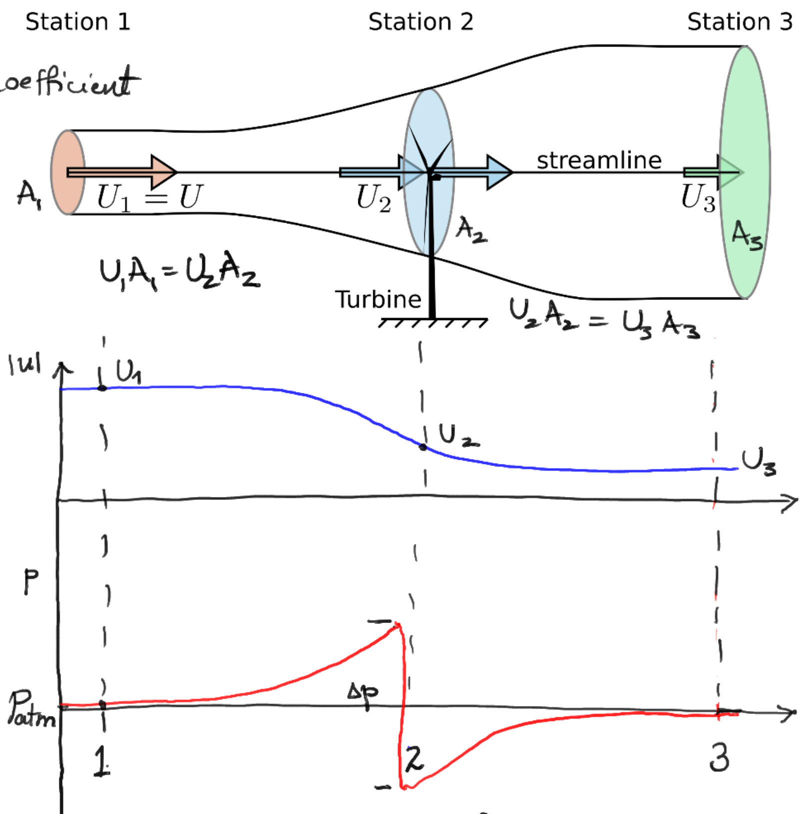
Application: Performance of a wind turbine

Given U, A and ρ.

Drag $D = \frac{1}{2}\rho U^2 A \times C_D$ ← drag coefficient

Power $P = \underbrace{\frac{1}{2}\rho U^3 A}_{\text{kinetic energy flux}} \times \eta$. ← efficiency

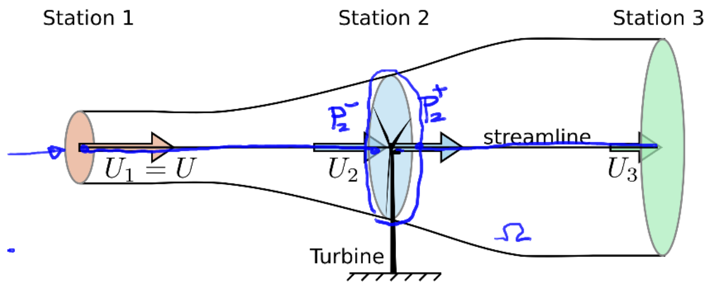
$\boxed{\eta(C_D)}$



$$\underline{p + \frac{1}{2}\rho|\underline{u}|^2} - \rho g \cancel{z}^{\,0} = \text{const.}$$

Mass conservation

$$U_1A_1 = U_2A_2 = U_3A_3 = Q.$$



Momentum balance (between 1 & 3).

$$(\rho U_1)(U_1A_1) - (\rho U_3)(U_3A_3) = +D. \quad \Rightarrow \quad D = \rho(U_1 - U_3)\,Q.$$

Energy conservation (between 1 & 3).

$$\left(\frac{1}{2}\rho U_1^2\right)(U_1A_1) - \left(\frac{1}{2}\rho U_3^2\right)(U_3A_3) = P. \quad \Rightarrow \quad P = \frac{1}{2}\rho(U_1^2 - U_3^2)\,Q.$$

Bernoulli eq^n between 1 & 2 — between 2 & 3

$$P_1 + \frac{1}{2}\rho U_1^2 = P_2^- + \underline{\frac{1}{2}\rho U_2^2} \qquad\qquad P_2^+ + \underline{\frac{1}{2}\rho U_2^2} = P_3 + \frac{1}{2}\rho U_3^2$$

$$P_2^- - P_2^+ = \left(\underset{P_{atm}}{\cancel{P_1}} + \frac{1}{2}\rho U_1^2\right) - \left(\underset{P_{atm}}{\cancel{P_3}} + \frac{1}{2}\rho U_3^2\right) = \frac{1}{2}\rho\,(U_1 + U_3)(U_1 - U_3).$$

$$D = (p_2^- - p_2^+)\,A_2 = \frac{1}{2}\rho\,(U_1+U_3)(U_1-U_3)\,A_2 \,.\; = \rho\,(U_1-U_3)\,A_2\,U_2\,.$$

$$\boxed{U_2 = \frac{U_1+U_3}{2}}$$

$$D = \rho\,(U_1-U_3)\,\overset{A_2U_2}{Q}\,. = \frac{1}{2}\rho U_1^2 A_2 \cdot C_D \;\Rightarrow\; C_D = 2\left(1-\frac{U_3}{U_1}\right)\frac{U_2}{U_1}\,.$$

$$P = \frac{1}{2}\rho\,(U_1^2-U_3^2)\,Q\,. = \frac{1}{2}\rho U_1^3 A_2\cdot\eta \;\Rightarrow\; \eta = \left(1-\frac{U_3^2}{U_1^2}\right)\cdot\left(\frac{U_2}{U_1}\right).$$

dimension-less

$U_2 = a U_1$, $U_3 = b U_1$.

a = turbine induction factor

b = wake induction factor.

$$C_D = 2(1-b)\,a$$
$$\eta = (1-b^2)\,a$$

$a = \frac{1+b}{2}$ or $b = 2a-1$.

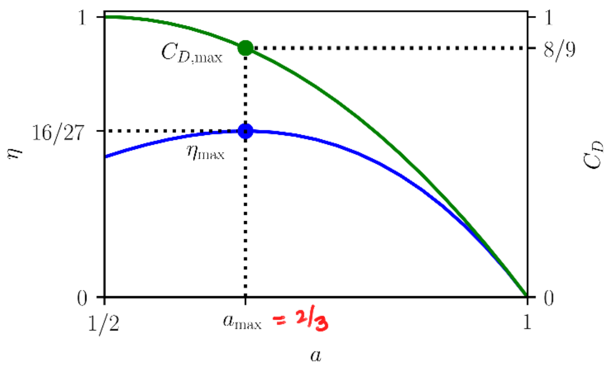
Aside:

$2(1-b)a = 2(1-2a+1)\,a = 4a(1-a)$

$2(1-b^2)a = 2\left[1-(1-2a)^2\right]a$

$= 2\left[1-1+4a-4a^2\right]a$

$= 8a^2(1-a)$,



$$\eta_{max} = \frac{16}{27} \approx 0.59 \;\ldots\; \text{Betz limit}$$

Betz-Joukowsky limit